18/07/2025 CONTINUOUS UNIFORM DISTRIBUTIONS Friday

*Rectangular (or) uniform distribution

Definition :- A continuous random variable "x" which all outcomes in a given range are equally likely in the interval a,b as the probability density function (p.d.f)

Ex: If the time taken by a bus to arrive is uniformly distributed between 0 and 10 minutes then any time between 0 and 10 minutes is equally likely. Here $a=0$, $b=10$ $= \frac{1}{10} = 0.1$.

$$f(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & ; \text{Otherwise} \end{cases}$$

is called uniform distribution with parameter a and b. It is usually denoted by $X \sim U(a,b)$.

***Distribution function:-** we know that

$$F(x) = P(X \le x) \quad ; a < x < b$$

$$= \int_a^x f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^x 1\,dx$$

$$= \frac{1}{b-a}\left[x\right]_a^x$$

$$\boxed{F(x) = \frac{x-a}{b-a}}$$

if $x=a$

$$F(x) = \frac{a-a}{b-a} = 0$$

if $x=b$

$$F(x) = \frac{b-a}{b-a} = 1$$

$$F(x) = \begin{cases} 0 & ; x \le a \\ \frac{x-a}{b-a} & ; a < x < b \\ 1 & ; x \ge b \end{cases}$$

Imp

* **Moments**:- we know that the r^{th} moment about origin is denoted by μ_r' and it is defined as $\mu_r' = E(x^r)$

$$\mu_r' = \int x^r f(x)\,dx$$

if $r=1$

$$\mu_1' = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$\boxed{\mu_1' = \frac{b+a}{2}}$$ — mean.

if $r=2$

$$\mu_2' = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$\mu_2' = \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]dx$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{\cancel{b-a}\,3}\right]$$

$$= \frac{1}{b-a}\left[\frac{(\cancel{b-a})(b^2+ab+a^2)}{3}\right] \quad \left[\because b^3-a^3 = (b-a)(b^2+a^2+ab)\right]$$

$$\boxed{\mu_2' = \frac{b^2+ab+a^2}{3}}$$

if $r=3$

$$\mu_3' = \int_a^b x^3 \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b x^3\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^4}{4}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^4-a^4}{4}\right]$$

$$= \frac{1}{b-a}\left[\frac{(b^2-a^2)(b^2+a^2)}{4}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{(\cancel{b-a})(b+a)(b^2+a^2)}{4}\right]$$

$$\boxed{\mu_3' = \frac{(b+a)(b^2+a^2)}{4}}$$

if $r=4$

$$\mu_4' = \int_a^b x^4 f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b x^4\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^5}{5}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^5-a^5}{5}\right]$$

$$\boxed{\mu_4' = \frac{b^5-a^5}{5(b-a)}}$$

*Central moments :-

$\boxed{\mu_1 = 0}$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$

$= \frac{4b^2+4ab+4a^2-3(a^2+b^2+2ab)}{12}$

$= \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12}$

$\boxed{\mu_2 = \frac{b^2+a^2-2ab}{12}}$ " — variance $\frac{(b-a)^2}{12}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= \frac{b+a\,(b^2+a^2)}{4} - 3\left[\frac{b^2+ab+a^2}{3}\right]\left[\frac{b+a}{2}\right] + 2\left[\frac{b+a}{2}\right]^3$

$= \frac{b^3+a^2b+ab^2+a^3}{4} - \frac{(b^3+ab^2+ab^2+a^2b+a^2b+a^2)}{3}$

$+ \frac{2(b^3+3a^2b+3b^2a+a^3)}{8}$

$= \frac{2b^3+2a^2b+2ab^2+2a^3-4b^3-8ab^2-8a^2b-4a^2+2b^3+6a^2b+6b^2a+2a^3}{8}$

$\boxed{\mu_3 = 0}$ "

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= \frac{b^5-a^5}{5(b-a)} - 4\left[\frac{(b+a)(b^2+a^2)}{4}\right]\left[\frac{b+a}{2}\right] + 6\left[\frac{b^2+ab+b^2}{3}\right]$

$\left[\frac{b+a}{2}\right]^2 - 3\left[\frac{b+a}{2}\right]^4$

$$\mu_4 = \frac{b^5 - a^5}{5(b-a)} - \frac{(b+a)^2(b^2+a^2)}{2} + \frac{3(b+a)^4}{8} - \frac{3(b+a)^4}{16}$$

Now simplify. last two terms

$$\frac{3(b+a)^4}{8} - \frac{3(b+a)^4}{16} = \frac{6(b+a)^4}{16} - \frac{3(b+a)^4}{16}.$$

$$= \frac{3(b+a)^4}{16}$$

final Expression

$$\mu_4 = \left[\frac{b^5 - a^5}{5(b-a)}\right] - \frac{(b+a)^2(b^2+a^2)}{2} + \frac{3(b+a)^4}{16}$$

Skewness:- β_1

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

∴ Rectangular distribution is a Symmetrical form

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

Kurtosis:- γ_2

$$\gamma_1 = \sqrt{\beta_1} = 0$$

$$\gamma_2 = \beta_2 - 3 = \frac{\mu_3^2}{\mu_2^3} - 3$$

imp

* Moment Generating function of uniform distributio

we know that M.G.F is denoted by $M_x(t)$ & is defined as $M_x(t) = E[e^{tx}]$

$$= \int_a^b e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{tb} - e^{ta}}{t}\right]$$

$$\boxed{M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$$ //

Imp

* characteristic function of uniform distribution:-

we know that C.F is denoted by $\phi_x(t)$ and is defined as $\phi_x(t) = E[e^{itx}]$

$$= \int_a^b e^{itx} f(x)\,dx$$

$$= \int_a^b e^{itx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{itx}}{it}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{itb} - e^{ita}}{it}\right]$$

$$\boxed{\phi_x(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}}$$ //

* CGF of uniform distribution:- we know that

C.G.F is denoted by $K_X(t)$ and is defined as

$$K_X(t) = \log M_X(t)$$

$$= \log\left[\frac{e^{tb}-e^{ta}}{t(b-a)}\right]$$

$$\boxed{K_X(t) = \log e^{tb} - \cancel{\log} e^{ta} - \log t(b-a)}$$,,

imp

* Mean deviation about mean of uniform distribution:-

we know that M.D about mean is defined as

$$\text{M.D about Mean} = E[|X - \text{Mean}|]$$

$$= \int_a^b |x - \text{Mean}|\, f(x)\, dx$$

$$= \int_a^b \left|x - \left(\frac{b+a}{2}\right)\right| \frac{1}{b-a}\, dx$$

let $t = x - \left(\frac{b+a}{2}\right)$

diff on both sides

$$dt = dx$$

if $x = a \Rightarrow a - \left(\frac{b+a}{2}\right) = \frac{2a-b-a}{2} = \frac{a-b}{2} = \frac{-(b-a)}{2}$

if $x = b \Rightarrow b - \frac{b+a}{2} = \frac{2b-b-a}{2} = \frac{b-a}{2}$.

$$\Rightarrow \frac{1}{b-a}\int_{-\frac{(b-a)}{2}}^{\frac{b-a}{2}} |t|\, dt \qquad \left\{\int_{-a}^{a} |x|\,dx = 2\int_0^a x\,dx\right\}$$

$$= \frac{1}{b-a}\, 2\int_0^{\frac{b-a}{2}} t\cdot dt \qquad \left[\because \text{where n is a Even function}\right]$$

$$= \frac{2}{b-a}\left[\frac{t^2}{2}\right]_0^{\frac{b-a}{2}}$$

$$= \frac{2}{b-a}\left[\frac{\left(\frac{b-a}{4}\right)^2-0}{2}\right]$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{(b-a)^{\cancel{2}}}{4}$$

$\boxed{\text{M.D about Mean} = \frac{b-a}{4}}$ //

S.M.Shami